On the health of a vector field with $RA^2/6$ coupling to gravity

Mindaugas Karčiauskas* and David H. Lyth†
Department of Physics, Lancaster University, Lancaster LA1 4YB, UK

The coupling $RA^2/6$ of a vector field to gravity was proposed as a mechanism for generating a primordial magnetic field, and more recently as a mechanism for generating a statistically anisotropic contribution to the primordial curvature perturbation. In either case, the vector field's perturbation has both a transverse and a longitudinal component, and the latter has some unusual features which call into question the health of the theory. We calculate for the first time the energy density generated by the longitudinal field perturbations, and go on to argue that the theory may well be healthy in at least some versions.

arXiv:1007.1426v1 [astro-ph.CO] 8 Jul 2010

I. INTRODUCTION

As far as we can tell, the initial condition for the formation of structure in the universe is provided by the primordial curvature perturbation ζ, which is known to exist when cosmological scales start to come inside the horizon. In turn, ζ is supposed ultimately to originate during an early era of inflation, as the vacuum fluctuation of one or more bosonic fields. Until recently these were always taken to be scalar fields, but following the proposal of Dimopoulos [1] vector fields are receiving considerable attention [2–11]. If a vector field contributes, statistical anisotropy of ζ may be expected, providing a smoking gun whose form will be a powerful discriminator between different models for generating the curvature perturbation.

In order to give a significant contribution to the curvature perturbation over a wide range of scales, the spectrum of the perturbation in each relevant field should be nearly scale-invariant. Barring an unlikely cancellation, this requires inflation to be almost exponential (de Sitter) while cosmological scales are leaving the horizon. Near scale-invariance is then automatic for scalar fields that are light (mass much less than the Hubble parameter) with canonically kinetic terms and minimal coupling to gravity. For a vector field in contrast, those same requirements lead to a contribution to the spectrum $\mathcal{P}_\zeta$ going like k^3. This almost certainly makes the contribution negligible on cosmological scales [4], because $\mathcal{P}_\zeta$ is constrained to be at most of order 1 on the much smaller scale leaving the horizon at the end of inflation (to avoid excessive black hole production).

For a light vector field A_μ with the canonical kinetic term, one recovers scale invariance by introducing a non-minimal coupling $RA^2/6$ to gravity [3, 4]. The spectrum of each transverse mode is then the same as for a scalar field, while the spectrum of the longitudinal mode is twice as big. The latter feature generates statistical anisotropy of a distinctive form [4, 5].

A coupling $RA^2/6$ has also been invoked in a completely different context, namely the case when A_μ describes the electromagnetic field [12]. In that case, the perturbation generated during inflation becomes a primordial magnetic field which may be cosmologically significant.

The coupling $RA^2/6$ thus has at least two possible uses, but concerns have been raised about its health. They are about the longitudinal mode [4, 10, 11] and are of two kinds. The first concern is that the effective kinetic term of the longitudinal mode (taking into account the non-minimal coupling) is negative on sub-horizon scales during inflation [4, 10]. As a result, one suspects that the corresponding particles carry negative energy density, allowing them to be created from the vacuum which makes it unstable. Such creation is known to occur for a minimally coupled scalar field with a negative-sign canonical kinetic term; a scalar field with this property is called a ghost and is cosmologically unacceptable.

The second concern [11] applies only if the action of A_μ has a non-zero mass term. Taking

*Electronic address: m.karciauskas@lancaster.ac.uk
†Electronic address: d.lyth@lancaster.ac.uk

the inflationary expansion to be isotropic, the longitudinal mode function becomes singular at a certain epoch after inflation is over. Taking into account the anisotropy in the inflationary expansion, caused at some level by the homogeneous part of A_μ, further singularities occur at least in the evolution equations [11].

If A_μ describes the electromagnetic field the mass term is absent and there are no singularities. If instead A_μ is supposed to generate a contribution to the curvature perturbation we cannot simply drop the mass term, but we will see in Section V how it might be replaced by something more complicated so as to avoid the singularities. If the singularities do exist they indicate a breakdown of the linear evolution, and progress in understanding what is going on could be made only with a non-linear calculation which is not performed here. In the absence of such a calculation, we are free to assume that any singularities have a negligible effect; in other words, that the linearly-evolved quantities match before and after an epoch of singularity.

In this paper, we focus mostly on the possibility of generating a contribution to the curvature perturbation. In Section II we review known results. In Section III we give the contribution of the vacuum fluctuation to the energy density, and in Section IV we give the contribution of particles along with a discussion of the issue of particle creation from the vacuum. In Section V we mention ways of avoiding a mass term in the action of the field and we conclude in Section VI.

II. BRIEF REVIEW OF KNOWN RESULTS

A. Action

Except where stated the notation and conventions of [4] are adopted. We are considering the action

$$\begin{aligned} S &= \int d\eta d^3x\sqrt{-g}\mathcal{L} \\ \mathcal{L} &= \frac{1}{2}m_P^2 R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}M^2A^2 \qquad (1) \\ M^2 &\equiv m^2 + \frac{1}{6}R, \qquad (2) \end{aligned}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and we are ignoring the back-reaction, which means that the line element is

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}x_ix_j = a^2(\eta)\left(-1 + \delta_{ij}x_ix_j\right). \qquad (3)$$

The curvature scalar is

$$R = -6\left(\dot{H} + 2H^2\right) = 3\left(3P/\rho - 1\right)H^2, \qquad (4)$$

where P and ρ are the pressure and energy density of the dominant component of the universe. During exponential inflation $R = -12H^2$. Going from the coordinate-induced basis to an orthogonal basis we arrive at the physical vector field $B_\mu = A_\mu/a$. (This notation for B_μ and A_μ is the reverse of the one in [4].)

This action is supposed to hold during inflation. It was first invoked [12] to generate a primordial magnetic field, with $m = 0$ and A_μ the electromagnetic vector potential. Then it was invoked in [3, 4] to allow A_μ to generate a contribution to the curvature perturbation, which might be done through the curvaton mechanism [13] or one of its generalisations [14].

The kinetic term of the action is the only gauge-invariant expression (confining ourselves as usual to a term that is quadratic in spacetime derivatives of A_μ). Additional possibilities [4] exist if the kinetic term breaks gauge invariance. One might consider them on the ground that gauge invariance is broken by the coupling $RA^2/6$ and (if it exists) the mass term. But

they would not give the desired flat spectrum for δA and we therefore reject them in the present context.[#1]

B. Transverse and longitudinal modes

We write the vector field as a sum of the homogeneous part ant the perturbation, $A_\mu(\mathbf{x},t) = A_\mu(t) + \delta A_\mu(\mathbf{x},t)$. The time component of the homogeneous part $A_\mu(t)$ vanishes, and the physical space components satisfy

$$\ddot{B}_i(t) + 3H(t)\dot{B}_i(t) + m^2 B_i(t) = 0, \tag{5}$$

which is the same as for a scalar field with the mass-squared m^2. While the time component of the perturbation is related to the space components δA_i by a constraint equation.

In what follows we work with the Fourier components of $\delta A_i(t,\mathbf{x})$ defined by

$$A_{i\mathbf{k}}(t) = \int d^3x \delta A_i(t,\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{6}$$

(In some formulas we use the physical momentum $p \equiv k/a$.)$A_{i\mathbf{k}}(t)$ can be written as

$$A_{i\mathbf{k}}(t) = \sum_\lambda e_i^\lambda(\mathbf{k}) A_{\lambda\mathbf{k}}(t), \tag{7}$$

where e_i^λ are polarization vectors. With the z axis chosen to be along the $\mathbf{k}$ direction, the polarization vectors are

$$e^{\mathrm{L}} \equiv \frac{1}{\sqrt{2}}(1,i,0), \quad e^{\mathrm{R}} \equiv \frac{1}{\sqrt{2}}(1,-i,0) \quad \text{and} \quad e^{||} \equiv (0,0,1), \tag{8}$$

and we choose $e_i^*(\mathbf{k}) = e_i(-\mathbf{k})$ so that $A^*_{\lambda\mathbf{k}}(t) = A_{\lambda,-\mathbf{k}}(t)$ making $A_\lambda(t,\mathbf{x})$ real. (In [4] both of these relations had a minus sign.) The transverse components L and R satisfy the same equations and we will use $\lambda = \perp$ to denote either of them.

The vector field is quantized by promoting it to an operator in the Heisenberg picture:

$$\hat{A}_{\lambda\mathbf{k}}(t) = \hat{a}_\lambda(\mathbf{k}) A_{\lambda k}(t) + \hat{a}^\dagger_\lambda(-\mathbf{k}) A^*_{\lambda k}(t) \tag{9}$$

$$\left[\hat{a}_\lambda(\mathbf{k}), \hat{a}^\dagger_{\lambda'}(\mathbf{k}')\right] = (2\pi)^3 \delta(\mathbf{k}-\mathbf{k}') \delta_{\lambda\lambda'} \tag{10}$$

Around the epoch of horizon exit corresponding to $aH = k$, the vacuum fluctuation of each mode is converted to a classical perturbation. The perturbation is gaussian so that the only connected correlator is

$$\langle A_{\lambda\mathbf{k}} A^*_{\lambda'\mathbf{k}'} \rangle = (2\pi)^3 \delta_{\lambda\lambda'} \delta(\mathbf{k}-\mathbf{k}') \frac{2\pi^2}{k^3} \mathcal{P}_\lambda(t,k) \tag{11}$$

$$\frac{2\pi^2}{k^3} \mathcal{P}_\lambda(t,k) = |A_{\lambda k}(t)|^2. \tag{12}$$

The spectrum $\mathcal{P}_\lambda(t,k)$ determines the expectation value of $A^2_\lambda(t,\mathbf{x})$ which is also its spatial average:

$$\langle A^2_\lambda(t) \rangle = \int_0^\infty \frac{dk}{k} \mathcal{P}_\lambda(t,k). \tag{13}$$

[#1] In the context of generating a magnetic field one might want to consider them, because a flat spectrum for δA is then not particularly desirable as we see after Eq. (42).

The vacuum fluctuation of the physical transverse mode $B_\perp = A_\perp/a$ is the same as that of a scalar field with mass-squared m^2. Well before horizon exit the action is that of a harmonic oscillator:

$$S_\perp = (2\pi)^{-3}\int d\eta d^3k\frac{1}{2}\left(|\partial_\eta A_{\perp\mathbf{k}}|^2 - k^2|A_{\perp\mathbf{k}}|^2\right). \tag{14}$$

We choose the mode function

$$A_{\perp k}(\eta) = e^{-ik\eta}/\sqrt{2k}, \tag{15}$$

so that $\hat{A}_{\perp\mathbf{k}}$ describes massless particles with momentum and energy $p = k/a$, and we choose the vacuum state. This choice is practically mandatory, because an occupation number $n_k \gtrsim 1$ on cosmological scales would almost certainly generate too much positive pressure. To be precise, this pressure would exceed the inflationary pressure $-3M_{\rm P}^2H^2$ at the beginning of inflation, unless fewer than $\ln(M_{\rm P}/H)/2$ e-folds of inflation occur before the observable universe leaves the horizon [15, 16].

The evolution equation for $A_{\perp k}$ is

$$\partial_\eta^2 A_{\perp k} + \left(k^2 + a^2M^2\right)A_{\perp k} = 0. \tag{16}$$

Well after horizon exit its solution has constant phase which means that we have a classical field $A_{\perp\mathbf{k}}$. (To be precise, a measurement could create such a field.) Just a few Hubble times after horizon exit we can take $m^2 = 0$ and H constant. Then the spectrum of the physical field is

$$\mathcal{P}_{B_\perp}(k) = \frac{k^3}{2\pi^2}\frac{|A_{\perp k}|^2}{a^2} = \left(\frac{H}{2\pi}\right)^2. \tag{17}$$

Since the spatial gradient is now negligible, the subsequent evolution is given by Eq. (5). Taking H to be constant, and assuming also $m \ll H$, Eq. (5) gives $B_i \propto t^{-m^2/3H}$. Taking also $\dot{H}/H^2$ to be constant, this gives the spectrum evaluated at a time when all relevant scales have left the horizon:[#2]

$$\mathcal{P}_\perp(t,k) = \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{2\eta-2\epsilon}, \qquad \eta \equiv m^2/3H^2, \quad \epsilon \equiv -\dot{H}/H^2, \tag{18}$$

with ϵ and η constant. (Note that this η has nothing to do with the conformal time.)

The action for the longitudinal mode is [10]

$$S_{||} = (2\pi)^{-3}\int {\rm d}\eta\,{\rm d}^3k\frac{1}{2}M^2\left[\frac{\left|\partial_\eta A_{||\mathbf{k}}\right|^2}{p^2+M^2} - a^2\left|A_{||\mathbf{k}}\right|^2\right]. \tag{19}$$

As long as $m \ll H$, we see that in the sub-horizon regime where p is slowly varying $\tilde{A}_{||\mathbf{k}} \equiv \sqrt{2}A_{||\mathbf{k}}H/p$ has the action

$$S_{||} \simeq -(2\pi)^{-3}\int d\eta d^3k\frac{1}{2}\left(|\partial_\eta\tilde{A}_{||\mathbf{k}}|^2 - k^2|\tilde{A}_{||\mathbf{k}}|^2\right). \tag{20}$$

Except for the minus sign this is the same as for the transverse mode. We choose the same mode function for $\tilde{A}_{||k}$ as for $A_{\perp k}$ in Eq. (15), corresponding to

$$A_{||k} = \frac{p}{\sqrt{2}H}\frac{1}{\sqrt{2k}}e^{-ik\eta}, \tag{21}$$

[#2] The approximations H and $\dot{H}/H^2$ constant should be adequate during the 15 or so e-folds that occur while cosmological scales are leaving the horizon.

and we choose the vacuum state. It was presumed in [4] that $\hat{A}_{||\mathbf{k}}$ describes particles with energy $-(k/a)$ and we verify that in the next section. Then the vacuum choice is practically mandatory, because the argument given for the transverse mode holds, except that we deal with negative energy density and pressure, and it is the energy density not the pressure which would spoil inflation [4].

The evolution equation for $A_{||k}$ is [11]

$$\left[\partial_\eta^2 + aH\frac{2p^2}{p^2+M^2}\left(1+\frac{1}{2HM^2}\frac{d(M^2)}{dt}\right)\partial_\eta + a^2\left(p^2+M^2\right)\right]A_{||k} = 0. \tag{22}$$

The second term in the first round bracket is negligible during inflation, because then it is equal to $-\dot{H}/H^2$. If our action continues to hold after inflation, this term may become significant and if m is nonzero M^2 will rise through zero making this term singular.

During inflation with $|m^2| \ll H^2$ the term (p^2+M^2) passes through zero around the time of horizon exit, but $A_{||k}$ is regular there [4, 11] and so is [11] $\dot{A}_{||k}/(p^2+M^2)$. Taking $m^2 = 0$ and H constant, one finds soon after horizon exit the spectrum $\mathcal{P}_{||} = 2\mathcal{P}_\perp$. The subsequent evolution $A_{||k}$ is again given by Eq. (5) so that when cosmological scales have left the horizon

$$\mathcal{P}_{||}(t,k) = 2\mathcal{P}_\perp(t,k). \tag{23}$$

with $\mathcal{P}_\perp$ given by Eq. (18).

After inflation is over, δA_μ can generate a contribution to the curvature perturbation through the curvaton mechanism [13] or one of its generalisations [14]. Through the δN formula [2, 4, 17, 18], Eqs. (18) and (23) then give the spectrum, bispectrum etc. of this contribution. Because of the factor 2 in Eq. (23), they exhibit distinctive anisotropy [4, 5].

On the assumption that A_μ instead describes the electromagnetic field [12], the physical vector potential B_i becomes time-independent in the super-horizon regime, and its transverse component gives a time-independent primordial magnetic field $\mathbf{B}_{\rm mag} = \mathrm{curl}\,\mathbf{B}$. The flat spectrum of $A_{\perp\mathbf{k}}$ makes the spectrum of $\mathbf{B}_{\rm mag}$ go like p^2, which makes it perhaps difficult to generate a useful primordial magnetic field on cosmological scales, and one might prefer the spectrum of $A_{\perp\mathbf{k}}$ to go like p^{-2} [19] so that $\mathbf{B}_{\rm mag}$ has a flat spectrum. The longitudinal mode of the electromagnetic field in this scenario has not been mentioned in the literature.

III. ENERGY DENSITY: GENERAL EXPRESSION AND VACUUM FLUCTUATION

A. General expression

The interaction $RA^2/6$ means that we are not dealing with Einstein gravity, but we still define the energy-momentum tensor through the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = M_{\rm P}^{-2}T_{\mu\nu}. \tag{24}$$

The contribution of bosonic fields is

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}\right)}{\delta g^{\mu\nu}}. \tag{25}$$

Since we used the Robertson-Walker metric to deal with A_μ, we do the same to evaluate the contribution of A_μ to Eq. (25). It is [8]

$$T_\mu^\nu = T_{1\mu}^{\ \nu} + T_{2\mu}^{\ \nu}, \tag{26}$$

where $T_{1\mu}^{\ \nu}$ is the part of the energy-momentum tensor which has the same form as the minimally coupled vector field

$$T^{\nu}_{1\mu} = \frac{1}{4}\delta^{\nu}_{\mu}F^2 - F_{\mu\kappa}F^{\nu\kappa} - \frac{1}{2}M^2\left(\delta^{\nu}_{\mu}A^2 - 2A_\mu A^\nu\right), \tag{27}$$

while $T^{\nu}_{2\mu}$ is an additional term due to non-minimal coupling to gravity

$$T^{\nu}_{2\mu} = \frac{1}{6}\left(R^{\nu}_{\mu} + \delta^{\nu}_{\mu}\nabla_\kappa\nabla^\kappa - \nabla_\mu\nabla^\nu\right)A^2. \tag{28}$$

The latter will become negligible at late times, when R becomes negligible.

The contribution of the unperturbed physical field to the energy density and pressure are the same as for a scalar field [8]:

$$\rho = \frac{1}{2}\left(\dot{B}^2 + \frac{1}{2}m^2B^2\right), \qquad P = \frac{1}{2}\left(\dot{B}^2 - \frac{1}{2}m^2B^2\right). \tag{29}$$

(They satisfy the continuity equation $\dot{\rho} = -3H(\rho + P)$ by virtue of the field equation (5).) The anisotropic stress is of the same order as the pressure, which when inserted into the Einstein equation would be inconsistent with the assumed Robertson-Walker metric. We therefore assume that the contribution of the unperturbed vector field to $T_{\mu\nu}$ is negligible. (Instead, one can invoke many randomly oriented vector fields, to arrive at a vector inflation model [8, 9].)

B. Vacuum fluctuation

Now we consider for the first time the contribution of δA to $T_{\mu\nu}$. We will take the spatial average which kills the anisotropic stress and gives ρ and P as a mode sum. For ρ we write

$$\rho(t) = \int_0^\infty \frac{dk}{k}\mathcal{P}_\rho(t,k), \qquad \mathcal{P}_\rho = 2\mathcal{P}_{\rho\perp} + \mathcal{P}_{\rho||}, \tag{30}$$

where the superscripts indicate transverse and longitudinal contributions to the spectrum. Then we write

$$\mathcal{P}_{\rho\perp} = \mathcal{P}_{\rho1\perp} + \mathcal{P}_{\rho2\perp}, \tag{31}$$

and similarly for $\mathcal{P}_{\rho||}$, where the first term comes from Eq. (27) and the second comes from Eq. (28), and we treat the pressure P in the same way.

In this section we consider the contributions generated by the vacuum fluctuation. The transverse contributions to the energy density are

$$\mathcal{P}_{\rho1\perp} = \frac{ap^3}{(2\pi)^2}\left[\left|\dot{A}_{\perp k}\right|^2 + \left(p^2 + M^2\right)|A_{\perp k}|^2\right], \tag{32}$$

$$\mathcal{P}_{\rho2\perp} = \frac{ap^3}{(2\pi)^2}\left[\left(3H^2 + \dot{H}\right)|A_{\perp k}|^2 - H\left(\dot{A}_{\perp k}A^*_{\perp k} + A_{\perp k}\dot{A}^*_{\perp k}\right)\right]. \tag{33}$$

And the longitudinal contributions are

$$\mathcal{P}_{\rho1||} = \frac{ap^3}{(2\pi)^2}\left[\frac{M^2}{p^2 + M^2}\left|\dot{A}_{||k}\right|^2 + M^2\left|A_{||k}\right|^2\right], \tag{34}$$

$$\begin{aligned}\mathcal{P}_{\rho2||} = \frac{ap^3}{(2\pi)^2}\Bigg[&-\frac{\left(7H^2 + \dot{H} + \frac{2H}{M^2}\frac{d(M)^2}{dt}\right)p^2}{\left(p^2 + M^2\right)^2}\left|\dot{A}_{||k}\right|^2 + \left(3H^2 + \dot{H}\right)\left|A_{||k}\right|^2 \\ &-H\frac{2p^2 + M^2}{p^2 + M^2}\left(\dot{A}_{||k}A^*_{||k} + A_{||k}\dot{A}^*_{||k}\right)\Bigg].\end{aligned} \tag{35}$$

The contributions to the pressure are

$$\mathcal{P}_{P1\perp} = \frac{ap^3}{12\pi^2}\left[\left|\dot{A}_{\perp k}\right|^2 + \left(p^2 - M^2\right)|A_{\perp k}|^2\right], \tag{36}$$

$$\begin{aligned}\mathcal{P}_{P2\perp} &= \frac{ap^3}{12\pi^2}\Big[2\left|\dot{A}_{\perp k}\right|^2 - 3\left(H^2+\dot{H}\right)|A_{\perp k}|^2 - \\ &\quad -2H\left(\dot{A}_{\perp k}A^*_{\perp k} + A_{\perp k}\dot{A}^*_{\perp k}\right) + \left(\ddot{A}_{\perp k}A^*_{\perp k} + A_{\perp k}\ddot{A}^*_{\perp k}\right)\Big].\end{aligned} \tag{37}$$

and

$$\mathcal{P}_{P1||} = \frac{ap^3}{12\pi^2}\left[\frac{M^2\left(3p^2M^2+M^2\right)}{\left(p^2+M^2\right)^2}\left|\dot{A}_{||k}\right|^2 - M^2\left|A_{||k}\right|^2\right], \tag{38}$$

$$\begin{aligned}\mathcal{P}_{P2||} &= -\frac{ap^3}{12\pi^2}\Bigg[\left(\frac{p^2L^2}{\left(p^2+M^2\right)^2} - 2\frac{2p^2+M^2}{p^2+M^2}\right)\left|\dot{A}_{||k}\right|^2 + \left(3H^2+3\dot{H}+2p^2\right)\left|A_{||k}\right|^2 \\ &+ \frac{p^2\left(11H + \frac{3}{M^2}\frac{dM^2}{dt}\right) + 2HM^2}{p^2+M^2}\left(\dot{A}_{||k}A^*_{||k} + A_{||k}\dot{A}^*_{||k}\right) - \left(\ddot{A}_{||k}A^*_{||k} + A_{||k}\ddot{A}^*_{||k}\right)\Bigg],\end{aligned} \tag{39}$$

where

$$L^2 \equiv 21H^2 - 7\dot{H} + 20\frac{H}{M^2}\frac{d(M)^2}{dt} + \frac{6}{M^4}\left(\frac{d(M)^2}{dt}\right)^2 - \frac{2}{M^2}\frac{d^2(M^2)}{dt^2}. \tag{40}$$

By virtue of the Eqs. (16) and (22), the continuity equation is satisfied by each mode separately. These expressions remain finite when p^2+M^2 goes through zero around the time of horizon exit, because $\dot{A}_{||k}/(p^2+M^2)$ remains finite. As discussed in the Introduction, we ignore the singularity that will occur after inflation if our action remains valid with nonzero m so that M^2 rises through zero.

Consider first the super-horizon regime $p^2 \ll H^2$, in which the vacuum fluctuation has generated a classical perturbation. Ignoring the possible epoch when M^2 rises through zero this is also the regime $p^2 \ll |M^2|$. If also $p \ll m$ the spatial gradient of δB_i is negligible and we can set $p = 0$ in Eqs. (32)–(35). Putting the result into Eq. (30), we arrive at the classical quantity $\rho_{\delta B}$. Using Eq. (13) and the analogous expression for $\langle|\delta\dot{B}|^2\rangle$, it can be written

$$\rho_{\delta B} = \frac{1}{2}\left(\langle|\delta\dot{B}|^2\rangle + m^2\langle|\delta B|^2\rangle\right). \tag{41}$$

This is of the same form as Eq. (29), and it holds at each position because the spatial gradient is negligible. The part of δB that comes from scales much bigger than the observable universe cannot be distinguished from the unperturbed quantity, and neither can its energy density.

If instead $m^2 = 0$ (or least negligible compared with p^2) we find to leading order in p^2

$$\mathcal{P}_{\rho\perp} = \frac{ap^3}{(2\pi)^2}p^2|A_{\perp k}|^2, \qquad \mathcal{P}_{\rho||} = 0. \tag{42}$$

Using Eqs. (12) and (30), this gives the known result $\rho_{\delta B} = \langle B^2_{\rm mag}/2\rangle$ where $\mathbf{B}_{\rm mag} = \mathrm{curl}\,\mathbf{B}$ is the magnetic field. We see that the longitudinal mode, ignored in previous work, gives in fact no contribution to ρ.

Now consider the regime $m \ll H$ and $p \gg H$. The latter condition (sub-horizon) means that $\hat{A}_{\perp\mathbf{k}}$ and $\hat{A}_{||\mathbf{k}}$ describe practically massless particles, and using Eqs. (15) and (21) and Eqs. (32)–(35) we find for the vacuum state

$$\mathcal{P}^{\rm vac}_{\rho\perp} = -\mathcal{P}^{\rm vac}_{\rho||} = \frac{p^4}{4\pi^2}. \tag{43}$$

These contributions to the energy density diverge in the ultra-violet and as usual we drop them.

IV. ENERGY DENSITY: PARTICLES

Instead of the vacuum state, suppose now that the state corresponds to occupation number n_k, which is independent of direction. The operator expression Eq. (12) and the mathematics of the harmonic oscillator imply that we should then make the replacement $\mathcal{P}^{\rm vac} \to 2n_k\mathcal{P}^{\rm vac}$. Then, remembering that the density of states is $1/(2\pi)^3$, Eq. (43) shows that each transverse particle carries energy p while each longitudinal particle carries energy $-p$. This holds for as long as $m \ll H$; in other words, unless and until we encounter the singularity epoch when M^2 rises through zero.

The negative energy density of the longitudinal mode occurs when the kinetic term in Eq. (19) is negative, which is caused by the non-minimal coupling to gravity. Since the longitudinal particles carry negative energy, energy conservation allows them to be created from the vacuum along with ordinary particles (and/or transverse particles) that carry positive energy.

A scalar field ϕ living in flat spacetime (minimal coupling to gravity), whose kinetic term is -1 times the canonical one, is called a ghost. Using flat spacetime quantum field theory, one can estimate the rate of processes like

$$\text{vacuum } \to \phi + \phi + \gamma + \gamma. \tag{44}$$

It is found [20] that the rate for this process would violate observational constraints on either air showers (assuming Lorentz invariance) or would require an implausibly low ultra-violet cutoff on the effective field theory (allowing a violation of Lorentz invariance).

The situation for our longitudinal mode is quite different. The non-minimal coupling to gravity, as well the condition $m \ll H$, mean that the creation rate cannot be calculated using flat spacetime theory. Also, unless m is strictly zero, the creation occurs only in the early universe which means that it is not constrained by direct observation.

After some particles have been created, their energy density redshifts like $1/a^4$ while they are relativistic, and like $1/a^3$ after they have become non-relativistic. The epoch of transition between these two regimes will be different for the longitudinal particles and the ordinary particles, which means that the initial cancellation between their energy densities will not be preserved. Much as in the warm inflation scenario, the energy density of the created particles will come into equilibrium, where the rate of creation balances the redshift. In order not to affect the creation of longitudinal perturbations from the vacuum, the longitudinal and transverse occupation numbers should be $n_k \ll 1$. (The presence of ordinary particles, giving zero total energy density, of course invalidates the argument given earlier for $n_k \ll 1$ being mandatory.)

It is not clear how to estimate the creation rate, which as already stated has nothing to do with flat spacetime field theory. Since the creation is caused by the coupling $RA^2/6$ though, significant creation will not persist until very late times even if m^2 remains negligible (so that the energy density of the longitudinal particles remains negative). This is because R becomes negligible at late times compared with all other relevant energy scales. In particular, Eq. (4) gives at the present epoch $|R| \sim H_0^2 \sim 10^{-66}\,\mathrm{eV}^2$. This means that longitudinal photon creation does not necessarily invalidate the use of our action as a mechanism for generating a primordial magnetic field, even though it implies in principle the existence of longitudinal photons.

V. AVOIDING A MASS TERM

After inflation, R given by Eq. (4) remains negative and at most of order H^2.[#3] A significant mass term will therefore cause M^2 to rise through zero at some point, leading to the singularities mentioned earlier. In this section we consider two ways in which such a term might be avoided.

[#3] With strict radiation domination $R = 0$, but non-Abelian interactions are expected to make R a significant fraction of H^2 even then [11].

The most promising way is to set $m=0$, but invoke a coupling $g^2\phi^2A^2$ or $g^2|\phi|^2A^2$, to some real or complex scalar field. The latter case, with the coupling coming from a gauge coupling, is particularly attractive, among other things because it justifies the canonical kinetic term. (Of course the gauge symmetry is broken by the $RA^2/6$ coupling, but that term disappears in the flat spacetime limit that will be an excellent approximation at late times.)#4 If ϕ has zero vev, such a coupling would allow the perturbation δA to affect the mass of ϕ, offering the possibility of an inhomogeneous decay mechanism for generating the curvature perturbation. Alternatively, if ϕ has nonzero vev this coupling could generate an effective mass for the vector field, which may avoid the singularity occurring when A_μ has a mass term and allow say a curvaton mechanism.

Another possibility might be to drop the coupling $RA^2/6$, and to introduce a negative mass-squared $m^2\simeq -2H^2$ during inflation. To stabilize A_μ one would then have to introduce a vector field potential $V(A^2)$ giving A^2 a vev at which $V(A^2)=0$, which if renormalizable would be of the form $V(A^2)=-m^2A^2+\lambda A^4$. This possibility (without an explicit form for $V(A^2)$) was invoked [1] to provide a curvaton mechanism. It might though turn out, when the field equation and energy density are worked out, that a singularity still develops when the effective mass-squared $dV/d(A^2)$ of the perturbation passes through zero.

Finally, we point out that instead of a single field A we might be dealing with a non-Abelian multiplet. Indeed, if A describes the electromagnetic field that is certainly the case, because A_μ that field is part of at least the electroweak $SU(2)\times U(1)$ multiplet which in turn may be part of a GUT multiplet. If A is a multiplet with (say) the $RA^2/6$ coupling to gravity, the issues we have raised will need to be revisited which has not been done. All that has been done in this direction (without reference to a specific mechanism for generating δA) is a calculation of the non-gaussianity of δA soon after horizon exit, that is generated by its self-coupling [7]. However, the self-interaction cannot be too strong, or the non-linear evolution of A_μ would not be viable. This places an upper bound on the gauge coupling, which is presently unknown and which might be violated by the Standard Model running couplings.

VI. CONCLUSION

A light vector field with the canonical kinetic term and minimal coupling to gravity is invariant under conformal transformations, and as a result it cannot contribute significantly to the primordial curvature perturbation (at least on cosmological scales). Nor can such a field (identified in that case as the electromagnetic vector potential) generate a viable primordial magnetic field.#5

To avoid this situation one must break the conformal invariance, by introducing a non-minimal kinetic term and/or a non-minimal coupling to gravity. We have explored the latter possibility, adopting the simplest viable coupling which is $RA^2/6$. That the factor $1/6$ works was discovered by accident [12], and it is not known whether the fact that it works is related at some deep level to the fact that a coupling $R\phi^2/6$ of a *scalar* field *restores* the otherwise broken conformal invariance.

In this paper we have addressed the issues, raised in [10, 11], that call into question the health of this coupling with regard to the longitudinal perturbation. To facilitate this, we have for the first time calculated the contribution of the vector field perturbation to the spatially-averaged energy density of the universe. We have verified that the contribution of the longitudinal component is negative unless and until the mass of the vector field becomes of order the Hubble parameter. Then longitudinal particles can at some level be created from the vacuum along with ordinary particles, but we have noted that such creation will become negligible at late times because the coupling of the field to gravity will be negligible.

#4 The view we are taking in this paper, that a gauge symmetry broken only by the $RA^2/6$ coupling may be relevant, is more up-beat than the one taken in Section 7.1 of [4].

#5 In fact, due to the conformal invariance there is no classical magnetic field at all on scales that have entered the horizon, because there is no Bogoliubov transformation.

In this regard, the longitudinal field is quite different from a scalar field which simply has the wrong-sign kinetic term; such a field is called a ghost and is indeed forbidden because it would lead to the creation of too many photons from the present-day vacuum [20].#6

We have also considered the other issue, which is that the linear evolution becomes singular at certain epochs if the action of the vector field has a nonzero mass term. We have pointed out that while this signals a failure of the linear theory, it remains to be seen whether the full theory is sick. In addition, we have mentioned ways in which a nonzero mass term can be replaced by something more complicated, which may well avoid the singularities.

VII. ACKNOWLEDGEMENTS

DHL and MK are supported by EU grant MRTN-CT-2006-035863 and DHL by EU grant UNILHC23792. MK also acknowledges the support of Lancaster University's Department of Physics.

---

[1] K. Dimopoulos, "Can a vector field be responsible for the curvature perturbation in the universe?," Phys. Rev. D **74** (2006) 083502.

[2] S. Yokoyama and J. Soda, "Primordial statistical anisotropy generated at the end of inflation," JCAP **0808** (2008) 005.

[3] K. Dimopoulos and M. Karčiauskas, "Non-minimally coupled vector curvaton," JHEP **0807** (2008) 119;

[4] K. Dimopoulos, M. Karčiauskas, D. H. Lyth and Y. Rodriguez, "Statistical anisotropy of the curvature perturbation from vector field perturbations," JCAP **0905** (2009) 013

[5] M. Karčiauskas, K. Dimopoulos and D. H. Lyth, "Anisotropic non-Gaussianity from vector field perturbations," Phys. Rev. D **80** (2009) 023509; C. A. Valenzuela-Toledo, Y. Rodriguez and D. H. Lyth, "Non-gaussianity at tree- and one-loop levels from vector field perturbations," Phys. Rev. D **80** (2009) 103519; C. A. Valenzuela-Toledo and Y. Rodriguez, "Non-gaussianity from the trispectrum and vector field perturbations," Phys. Lett. B **685** (2010) 120.

[6] K. Dimopoulos, M. Karčiauskas and J. M. Wagstaff, "Vector Curvaton with varying Kinetic Function," Phys. Rev. D **81** (2010) 023522; K. Dimopoulos, M. Karčiauskas and J. M. Wagstaff, "Vector Curvaton without Instabilities," Phys. Lett. B **683** (2010) 298;

[7] N. Bartolo, E. Dimastrogiovanni, S. Matarrese and A. Riotto, "Anisotropic bispectrum of curvature perturbations from primordial non-Abelian vector fields," JCAP **0910** (2009) 015; N. Bartolo, E. Dimastrogiovanni, S. Matarrese and A. Riotto, "Anisotropic Trispectrum of Curvature Perturbations Induced by Primordial Non-Abelian Vector Fields," JCAP **0911** (2009) 028; E. Dimastrogiovanni, N. Bartolo, S. Matarrese and A. Riotto, "Non-Gaussianity and statistical anisotropy from vector field populated inflationary models," arXiv:1001.4049 [astro-ph.CO].

[8] A. Golovnev, V. Mukhanov and V. Vanchurin, "Vector Inflation," JCAP **0806** (2008) 009;

[9] A. Golovnev, V. Mukhanov and V. Vanchurin, "Gravitational waves in vector inflation," JCAP **0811** (2008) 018; A. Golovnev and V. Vanchurin, "Cosmological perturbations from vector inflation," Phys. Rev. D **79** (2009) 103524.

[10] B. Himmetoglu, C. R. Contaldi and M. Peloso, "Instability of anisotropic cosmological solutions supported by vector fields," Phys. Rev. Lett. **102** (2009) 111301; B. Himmetoglu, C. R. Contaldi and M. Peloso, "Instability of the ACW model, and problems with massive vectors during inflation," Phys. Rev. D **79** (2009) 063517.

[11] B. Himmetoglu, C. R. Contaldi and M. Peloso, "Ghost instabilities of cosmological models with vector fields nonminimally coupled to the curvature," Phys. Rev. D **80** (2009) 123530.

[12] M. S. Turner and L. M. Widrow, "Inflation Produced, Large Scale Magnetic Fields," Phys. Rev. D **37** (1988) 2743.

[13] A. D. Linde and V. F. Mukhanov, "Nongaussian isocurvature perturbations from inflation," Phys. Rev. D **56** (1997) 535; D. H. Lyth and D. Wands, "Generating the curvature perturbation without an inflaton," Phys. Lett. B **524** (2002) 5; T. Moroi and T. Takahashi, "Effects of

#6 This difference is not recognised in footnote 16 of [11], which states that the mass zero case is ruled out by [20].

cosmological moduli fields on cosmic microwave background," Phys. Lett. B **522** (2001) 215 [Erratum-ibid. B **539** (2002) 303]; D. H. Lyth, C. Ungarelli, and D. Wands, "The primordial density perturbation in the curvaton scenario," Phys. Rev. D **67**, 023503 (2003).

[14] D. H. Lyth and A. R. Liddle, *The Primordial Density Perturbation (Cosmology, Inflation and the origin of Structure)* , Cambridge University Press, 2009.

[15] D. H. Lyth and A. Riotto, "Particle physics models of inflation and the cosmological density perturbation," Phys. Rept. **314** (1999) 1

[16] A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large-Scale Structure*, Cambridge University Press, 2000.

[17] A. A. Starobinsky, "Multicomponent De Sitter (Inflationary) Stages And The Generation Of Perturbations," JETP Lett. **42**, 152 (1985) [Pisma Zh. Eksp. Teor. Fiz. **42**, 124 (1985)]; M. Sasaki and E. D. Stewart, "A General analytic formula for the spectral index of the density perturbations produced during inflation," Prog. Theor. Phys. **95** (1996) 71;

[18] D. H. Lyth, K. A. Malik and M. Sasaki, "A general proof of the conservation of the curvature perturbation," JCAP **0505**, 004 (2005); D. H. Lyth and Y. Rodriguez, "The inflationary prediction for primordial non-gaussianity," Phys. Rev. Lett. **95** (2005) 121302.

[19] J. Martin and J. Yokoyama, "Generation of Large-Scale Magnetic Fields in Single-Field Inflation," JCAP **0801** (2008) 025.

[20] J. M. Cline, S. Jeon and G. D. Moore, "The phantom menaced: Constraints on low-energy effective ghosts," Phys. Rev. D **70** (2004) 043543.